THE ERRORS OF COMPENSATED GYROSCOPIC INSTRUMENTS WITH RANDOMLY VARYING VELOCITY ERROR

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We shall investigate the behavior of a navigational gyroscopic instrument (gyrocompass, gyrovertical) which moves arbitrarily on the earth's surface. If the instrument is autonomous, that is if it does not receive data from the motion of other instruments, then it can be made imperturbable (invariant with respect to the motion of the instrument's base) by a suitable design and a suitable selection of the values of its parameters.

It is known, however, that the error of an autonomous imperturbable navigational instrument caused by non-zero initial conditions does not die out. On the other hand, if the autonomy is not required, that is if we assume that the velocity of the base can be determined at any instant of time from the readings of other navigational instruments (more correctly, if we know the exact values of the velocity components along the geographic axes), then the condition of imperturbability does not determine uniquely the dynamic characteristics of a navigational instrument, which may then remain invariant with respect to the motion of its base even when free vibrations of arbitrary period are present and when the damping regime is also arbitrary, Glitscher [1], Ishlinskii, Roitenberg and his students, Bonder and Seleznev [2] and others investigated designs of imperturbable gyroscopic instruments in which the conditions of autonomous imperturbability were not satisfied.

That in principle it is possible to construct such an instrument can be argued as follows: Let the instrument have an arbitrary period and an arbitrary damping coefficient, and let us assume that both parameters are known. Then, knowing the velocity components of the base at any instant of time we can exactly calculate the ballistic error of the instrument. This calculation can be carried out by an independent computer.

Ideally exact operations performed by the computer would give corrections of the instrument's reading which in turn would yield the exact value of a navigational parameter independent both of the period and of the damping coefficient.

It is practically impossible to make a perfectly imperturbable instrument with arbitrary coefficients, for the following reasons:

1. The parameters of the instrument are known only approximately.

2. Velocity measurements involve errors.

3. The reduction of velocities in the computer also involves errors.

Consequently, the forced error component depending on parameters is unavoidable, and the selection of the parameters requires compromises. On one hand it is desirable that the period be as small as possible and damping be as large as possible in order to insure sufficiently fast damping of an error arising from non-zero initial conditions. On the other hand the period should not differ greatly from the period of Schuler, and the damping should be as small as possible in order to minimize the error arising from the violation of the conditions of imperturbability. The most convenient values of the parameters depend on the relationship between the different values of initial conditions and the errors in measuring and reducing velocities. In this work we present an analysis of the errors of the instrument and determination of the optimal values of the parameters.

We consider here the differential equation of motion of a single rotor gyrocompass with an arbitrary period and arbitrary damping.

It is known that the form of the system of differential equations controlling the behavior of a gyrocompass depends essentially on its rotation about the "North-South" axis. For a two-rotor damped gyrocompass stabilized about the "North-South" axis the equations of motion are of the sixth order. For a one rotor gyrocompass not stabilized about the "North-South" axis it is necessary, in general, to take into account the inertia of the suspension rings. This would raise the order of the system to six, like in the previous case. Satisfying the well known conditions of Schuler does not assure the imperturbability of a gyrocompass.

If we consider, however, the problem in which only the velocity reading error is taken into account then we can describe the properties of a gyrocompass through a simple system of linear differential equations of the second order with constant coefficients. Such a description is only a rough approximation and is causing loss of certain dynamic properties. Nevertheless this simplification (widely used in the past by Bulgakov and Nikolai, for example) is permissible and useful for the following reasons: In the first place the basic dynamic properties are preserved (oscillatory character of errors, possibility in theory of reducing the error to zero at zero initial conditions, dependence of errors on the variation of parameters, etc.), in the second place it is possible to obtain solutions in closed form. Bulgakov [3] used this simplification when he solved a problem on the accumulation of ballistic deviations.

In the case of a more complicated and more realistic model this idealization enables us to investigate separately the influence of only one factor, that is of the imperfection of the velocity meter. This means that we can estimate the errors in the case when other conditions, besides the velocity meter, are ideal, (the absence of rocking is equivalent to a perfect stabilization about the "North-South" axis with respect to other instruments, the inertia of gimbal rings can be neglected).

1. The expressions for errors of a gyroscopic instrument with an arbitrary period, arbitrary damping and with random variation of the velocity error. We shall assume that ballistic errors are corrected from the readings of an outside imperfect velocity meter. In this case the equation of motion is linear and on the right-hand side we shall have in the role of perturbation the error of the velocity meter. The equation of motion of the gyroscope has the form

$$\frac{d^2}{dt^2}(\alpha-\alpha^*)+\varepsilon\frac{Ql}{H}\frac{d}{dt}(\alpha-\alpha^*)+\frac{Ql}{H}\Omega\cos\varphi(\alpha-\alpha^*)=$$
$$=\varepsilon\left(\frac{Ql}{H}\right)^2\frac{1}{g}\frac{d\Delta V_N}{dt}+\frac{Ql}{H}\left(\frac{1}{R}-\Omega\cos\varphi\frac{Ql}{gH}\right)\Delta V_N \qquad (1.1)$$
$$\alpha^*=\frac{Ql}{gH}V_N \qquad (1.2)$$

Here α is the angle between the rotor axis and the plane of the meridian, α^* is the velocity deviation, V_N is the northern velocity component of the base, ΔV_N is the error arising in the reading of V_N and in the reduction of V_N. Ql is the static moment of the inner frame, H is the angular momentum of the rotor, R is the earth's radius, ε is the parameter of damping. In deriving the equation it has been assumed that $V \ll R\Omega\cos\varphi$ which indicates that the western velocity component can be neglected. When the conditions of Schuler are satisfied ($\varepsilon = 0$, $R\Omega Ql\cos\varphi = gH$) the instrument becomes imperturbable. The error ΔV_N will be treated as a random function of time with known statistical properties, namely we shall assume that the error ΔV_N is stationary, its mathematical expectation equals zero and the correlation function is

$$R_V(\Delta t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Delta V_N(t+\Delta t)\,\Delta V_N(t)\,dt=\sigma^2\rho'_V(\Delta t)\quad(\rho'_V(0)=1) \qquad (1.3)$$

It is implied here that ΔV_N is independent of α, which is a good first approximation when the velocity V_N is sufficiently small. The initial conditions are random quantities with zero mathematical expectation and with correlation moments*

$$\langle[\Delta\dot{\alpha}(0)]^2\rangle = B_1^2, \qquad \langle[\Delta\dot{\alpha}(0)]^2\rangle = B_2^2,$$

$$\langle[\Delta\dot{\alpha}(0)\,\Delta\dot{\alpha}(0)]\rangle = B_{12}^2 \qquad (\Delta\alpha = \alpha - \alpha^*) \tag{1.4}$$

We shall introduce new variables

$$x = \frac{R\Omega\cos\varphi}{\sigma}\Delta\alpha, \qquad \tau = \sqrt{\frac{g}{R}}\,t, \qquad v = \frac{\Delta V_N}{\sigma} \tag{1.5}$$

Then Equation (1.1) becomes

$$\frac{d^2x}{d\tau^2} + 2\xi\frac{dx}{d\tau} + \omega_0^2 x = \omega_0^2\left[2\xi\frac{dv}{d\tau} + (1-\omega^2{}_0)\,v\right]$$

$$\left(2\xi = \varepsilon\frac{Ql}{H}\sqrt{\frac{R}{g}}, \quad \omega_0^2 = \frac{QlR\Omega\cos\varphi}{gH}\right) \tag{1.6}$$

For the initial conditions we have

$$\langle[x(0)]^2\rangle = C_1^2, \quad \langle[(dx\,|\,d\tau)_{\tau=0}]^2\rangle\ C_2^2, \quad \langle[x(0)\,(dx\,|\,d\tau)_{\tau=0}]\rangle = C_{12}^2 \tag{1.7}$$

where

$$C_1 = B_1\frac{R\Omega\cos\varphi}{\sigma}, \qquad C_2 = B_2\frac{R^2\Omega\cos\varphi}{\sigma g}, \qquad C_{12} = B_{12}\frac{R^{3/2}\Omega\cos\varphi}{\sigma g^{1/2}}$$

The correlation function for v equals

$$\rho_{V'}(\tau) = \rho'_V\left(\sqrt{R/g}\,\Delta t\right) \tag{1.8}$$

Let us mention that the equation of a gyroscopic pendulum with one rotor (gyrovertical) can be reduced to (1.6). The solution of Equation (1.6) with the initial conditions

$$x = x_0, \qquad dx/d\tau = \dot{x}_0 \qquad \text{for } \tau = 0$$

with an arbitrary $v(\tau)$ and at $\xi < \omega_0$ (oscillatory case) has the form

$$x(\tau) = e^{-\xi\tau}\left(x_0\cos\sqrt{\omega_0^2-\xi^2}\,\tau + \frac{\dot{x}_0+\xi x_0}{\sqrt{\omega_0^2-\xi^2}}\sin.\sqrt{\omega_0^2-\xi^2}\,\tau\right) +$$

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* Here and from now on the mathematical expectations and moments will be within brackets.

$$+\frac{\omega_0^2}{\sqrt{\omega_0^2-\xi^2}}\int_0^\tau\Big\{2\xi e^{-\xi t}\sin\sqrt{\omega_0^2-\xi^2}\,t\,\frac{d}{d\tau}v(\tau-t)+$$
$$+(1-\omega_0^2)\,e^{-\xi t}\sin\sqrt{\omega_0^2-\xi^2}\,t\,v(\tau-t)\Big\}\,dt \tag{1.9}$$

After simple transformations we obtain

$$x(\tau)=e^{-\xi\tau}\left(x_0\cos\sqrt{\omega_0^2-\xi^2}\,\tau+\frac{\dot{x}_0+\xi x_0+2\xi\omega_0^2v(0)}{\sqrt{\omega_0^2-\xi^2}}\sin\sqrt{\omega_0^2-\xi^2}\,\tau\right)+$$
$$+\omega_0^2\int_0^\tau e^{-\xi t}\left[2\xi\cos\sqrt{\omega_0^2-\xi^2}\,t+\frac{1-\omega_0^2-2\xi^2}{\sqrt{\omega_0^2-\xi^2}}\sin\sqrt{\omega_0^2-\xi^2}\,t\right]v(\tau-t)\,dt \tag{1.10}$$

To make it simple we shall regard $v(0) = 0$. Assuming that the initial conditions are statistically independent of $v(\tau)$ we obtain the value of the dispersion of error at any instant of time τ by taking the average over the whole spectrum (see, for example [4])

$$\overline{[x(\tau)]^2}=I_1+I_2 \tag{1.11}$$

where

$$I_1=e^{-2\xi\tau}\left\{\frac{C_1^2+2\xi C_{12}^2+\xi^2C_2^2}{\omega_0^2-\xi^2}\sin^2\sqrt{\omega_0^2-\xi^2}\,\tau+C_1^2\cos^2\sqrt{\omega_0^2-\xi^2}\,\tau+\right.$$
$$\left.+\frac{2(C_{12}^2+\xi C_2^2)}{\sqrt{\omega_0^2-\xi^2}}\sin\sqrt{\omega_0^2-\xi^2}\,\tau\cos\sqrt{\omega_0^2-\xi^2}\,\tau\right\}$$

$$I_2=\omega^4{}_0\int_0^\tau e^{-\xi\tau_1}\left\{2\xi\cos\sqrt{\omega_0^2-\xi^2}\,\tau_1-\frac{1-\omega_0^2-2\xi^2}{\sqrt{\omega_0^2-\xi^2}}\sin\sqrt{\omega_0^2-\xi^2}\,\tau_1\right\}\times$$
$$\times\int_0^\tau e^{-\xi\tau_2}\left\{2\xi\cos\sqrt{\omega_0^2-\xi^2}\,\tau_2-\frac{1-\omega_0^2-2\xi^2}{\sqrt{\omega_0^2-\xi^2}}\sin\sqrt{\omega_0^2-\xi^2}\,\tau_2\right\}\rho_V(\tau_1-\tau_2)\,d\tau_2\,d\tau_1 \tag{1.12}$$

If $\xi = \omega_0$ (critical damping), then instead of (1.10) we have

$$x(\tau)=x_0e^{-\omega_0\tau}+(\dot{x}_0+x_0\omega_0)\,\tau e^{-\omega_0\tau}+$$
$$+\omega_0^2\int_0^\tau\{-2\omega_0-(3\omega_0^2-1)\,t\}\,e^{-\omega_0 t}\,v(\tau-t)\,dt \tag{1.13}$$

After similar transformations instead of (1.12) we obtain

$$I_1=e^{-2\omega_0\tau}\{C_1^2+2(C_{12}^2+\omega_0C_1^2)\,\tau+(C_2^2+2\omega_0C_{12}^2+\omega_0^2C_1^2)\,\tau^2\}$$

$$I_2=\omega_0^4\int_0^\tau[-2\omega_0+(3\omega_0^2-1)\tau_1]\,e^{-\omega_0\tau_1}d\tau_1\int_0^\tau\Big[-2\omega_0+$$

$$+ (3\omega_0^2 - 1)\tau_2\Big] e^{-\omega_0\tau_2} \rho_V(\tau_1 - \tau_2)\, d\tau_2 \tag{1.14}$$

Similarly, we can obtain the expression for the dispersion of error when damping is greater than critical. This case, however, is of no interest in practice. The obtained formulas permit us to solve the analytical part of the problem, which is the calculation of the mean square error at any instant of time when the parameters are known.

2. Finding the optimum values of the parameters of a gyro-instrument. After determining the dispersion of errors we can start on the problem of selecting parameters ω_0 and ξ and of finding the smallest mean square error at any given instant of time $\tau = T$. The conditions for the extremum are

$$\frac{\partial (I_1 + I_2)}{\partial \xi} = 0, \qquad \frac{\partial (I_1 + I_2)}{\partial \omega_0} = 0 \tag{2.1}$$

We can in principle, easily obtain from the above equations the optimal values of $\omega_0 = \omega_0'$ and of $\xi = \xi'$. In practice, however, the solution becomes involved because I_1 and I_2 have different values when $\xi < \omega_0$, when $\xi = \omega_0$, and when $\xi > \omega_0$. The solution has to be searched in the following order:

1. Solve Equations (2.1) where the values of I_1 and I_2 are those from Expressions (1.12) for the case when $\xi < \omega_0$. If this will result in $0 < \xi' < \omega_0'$, then the solution is not contradictory. If not, then the assumption that $\xi' < \omega_0'$ is obviously wrong.

2. Solve Equations (2.1) where the values of I_1 and I_2 are those for $\xi > \omega_0$. If the assumption that $\xi > \omega_0$ is correct then the result should be $0 < \omega_0' < \xi'$. If not, then the third possibility remains.

3. Solve Equation (2.1) where the values of I_1 and I_2 are those from (1.14) and determine $\xi' = \omega_0'$.

It is reasonable to expect that in each case the solution is unique. If, however, instead of one value we obtain more extremal values for the error, then they have to be compared with each other and the smallest one selected.

This procedure leads to very long calculations, because the equations are quite complicated. In order to obtain closed solutions we shall introduce certain additional simplifications whose justification is given below. If $\xi \ll 1$, then we can expect $\omega_0' \approx 1$, because the more ω_0 differs from unity the greater is the absolute value of the right-hand side of (1.6), and, consequently, the greater is the dispersion of the forced component of the error. At the same time the velocity of damping of the error arising from the initial conditions, on the average, does not depend on ω_0 when $\omega_0 \gg \xi$. This means that in this case the value $\omega_0 = 1$ should be close to the optimal value.

Assuming that $\xi < 1$ when $\omega_0 = 1$ may lead to a contradiction, meaning that the equations would give $\xi' > 1$, hence $\omega_0 > 1$. Besides, it is seen that the critical case $\omega_0 = \xi$ is close to the optimal one because with further increase of ξ (with damping greater than the critical) the intermediate process also slows down and the dispersion of the forced component increases. Therefore the case $\xi > \omega_0$ is of no practical interest.

We shall consider now two cases: the case of small damping $\omega_0' = 1$, $\xi' < 1$ and the case of the critical damping $\omega_0' = \xi' > 1$.

Depending on the values of the two principal parameters, that is of the relative magnitude of the initial conditions and of the time T we either leave the period equaling the period of Schuler, unchanged, or we decrease the period making the damping critical. Both variants are analyzed in detail in (3) and (4) of this paper. To be specific, we shall assume from now on that the correlation function of the error v is

$$\rho_V(\Delta\tau) = e^{-b|\Delta\tau|} \tag{2.2}$$

To make it simpler we shall consider $b \gg 1/T$. This expresses the assumption that the interval of correlation of the error is considerably smaller than the considered intervals of time.

3. The case of small damping. Retaining only first order terms with respect to $1/b$ when $\omega_0 = 1$ we obtain

$$I_2 = \frac{2}{b}\left\{\xi + e^{-2\xi T}\left[\frac{\xi}{1-\xi^2} + \frac{\xi^3}{1-\xi^2}\cos 2\sqrt{1-\xi^2}\,T + \right.\right.$$
$$\left.\left. + \frac{\xi^2}{\sqrt{1-\xi^2}}\sin 2\sqrt{1-\xi^2}\,T\right]\right\} \tag{3.1}$$

The first equation in (2.1) determines ξ. The results can be obtained in closed form if we make the following assumptions:

1. The initial phase is equally probable, that is

$$C_{12} = 0, \qquad C_1^2 = C_2^2 = 2D^2/b \qquad (D = \text{const}) \tag{3.2}$$

2. The initial conditions are sufficiently large, that is

$$D^2 \gg 1 \tag{3.3}$$

Then, approximately

$$\langle [x(T)]^2 \rangle \approx 2b^{-1}(\xi + D^2 e^{-2\xi T}) \tag{3.4}$$

From the first condition for minimum (2.1) with respect to ξ we obtain

$$\xi' = (1/2T)\ln 2TD^2 \tag{3.5}$$

Depending on the values of T and D^2 we obtain from (3.5) that $\xi' \leqslant 1$ or $\xi' > 1$.

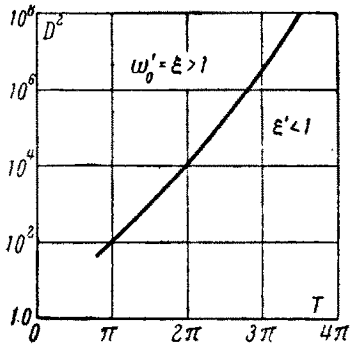

Fig. 1.

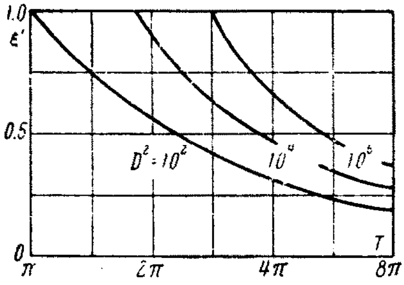

Fig. 2.

On the T, D^2 plane the regions where the above inequalities are satisfied are separated by the curve

$$(1/2T)\ln 2TD^2 = 1$$

For the region to the right of the above curve (Fig. 1) the initial hypothesis ($\xi' < 1$) is correct; under these conditions it is convenient to keep $\omega_0' = 1$. On the other hand, for the region to the left of the curve, as shown previously, we have to make $\omega_0' = \xi' > 1$.

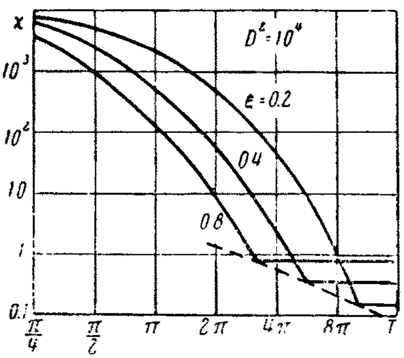

Fig. 3.

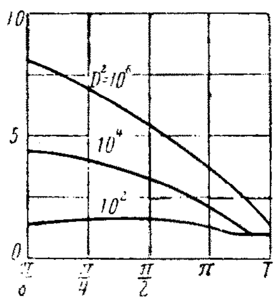

Fig. 4.

Let us consider the case of small damping. At any fixed instant of time there is a corresponding value of ξ. Figure 2 shows the curves

$\xi' = \xi'(T)$ at $D^2 = 10^2$, 10^4, 10^6. The optimal value of ξ at a given fixed instant of time T is not optimal at another instant of time. In Fig. 3 are given the curves

$$\chi = f(T) \qquad \left(\chi = \frac{b}{2}\langle[x(T)]^2\rangle\right) \qquad \text{when } D^2 = 10^4 \tag{3.6}$$

As expected, at large values of ξ the error decreases faster, until it attains a stationary value. The equation of the envelope of the family of curves (shown in Fig. 3 by a dotted line), obtained by substituting (3.5) in (3.4), is

$$\frac{b}{2}\langle[x(T)]^2\rangle\min = \frac{1}{2T}(1 + \ln 2TD^2) \tag{3.7}$$

This curve determines the limiting possibilities of a system with constant linear damping. Let us mention that the main qualitative result of this article is that the damping should be smaller for larger time intervals because of the random variation of the velocity error. If we assume the most disadvantagous variations as done by Bulgakov [3], then, for all practical purposes, the optimal damping would be the critical damping.

4. The case of critical damping. Assuming (3.2) and (3.3) we obtain the following approximate expression for the dispersion of error at the instant of time $\tau = T$

$$\langle[x(T)]^2\rangle = \frac{2}{b}\left[\frac{5}{4}\omega_0^5 + D^2[(1+\omega_0 T)^2 + T^2]\, e^{-2\omega_0 T}\right] \tag{4.1}$$

The condition for the minimum with respect to ω_0 is

$$\frac{25\omega_0^4 e^{2\omega_0 T}}{2T^2D^2(\omega_0^2 T + \omega_0 + T)} = 1 \tag{4.2}$$

The results of a numerical solution of the above equation are shown in Fig. 4 in the form of the curves $\omega_0' = \omega_0'(T)$ at $D^2 = 10^2$, 10^4, 10^6.

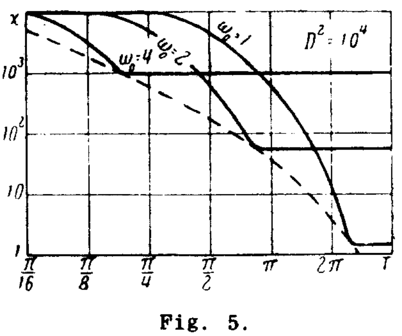

Fig. 5.

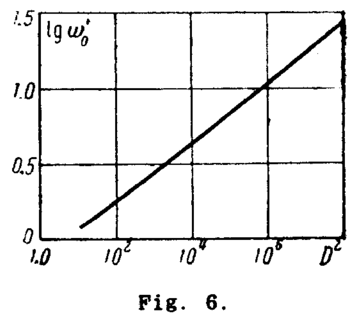

Fig. 6.

As in the previous case each value turns out to be optimal only at a given T. The curves (3.6) are shown in Fig. 5. The envelope of the family of these curves (shown in Fig. 5 by a dotted curve) is obtained by substituting (4.2) in (4.1). From the curves in Figs. 3 and 5 we can also determine how critical is our system with respect to the variation of the parameters ξ and ω_0, that is for which intervals of time the selected values of parameters give results sufficiently close to the optimal results. Figure 4 shows that even for wide bounds of variations of D^2 and T, the frequency ω_0 should not be too large. An estimate of the upper bound for ω_0 independent of T can be easily obtained directly from the Formula (4.1). Practical considerations indicate that the value of ω_0 should be selected such that at least

$$\langle [x(\infty)]^2\rangle / \langle [x(0)]^2\rangle < 1/2$$

Otherwise the mean square value of the initial error decreases less than twice. Hence we obtain

$$5\omega_0^5 < D^2$$

Figure 6 shows the curve $\omega_0 = \sqrt[5]{(D^2/5)}$ which is the upper limit for ω_0'.

The results of the investigations presented in (3) and (4) are plotted in the D^2, T plane. Figure 7 shows the curves of constant ratio

$$m^2 = \langle [x(0)]^2\rangle / \langle [x(T)]^2\rangle$$

that is of the ratio of the dispersion of the initial error to the dispersion of the error of the optimal system at the instant of time T.

The curves are drawn for m = 10 and for m = 100 (decrease of the mean square error 10 times and 100 times, respectively). This graph, then, gives the relationship between these parameters at which the desired decrease of the mean square error is obtained.

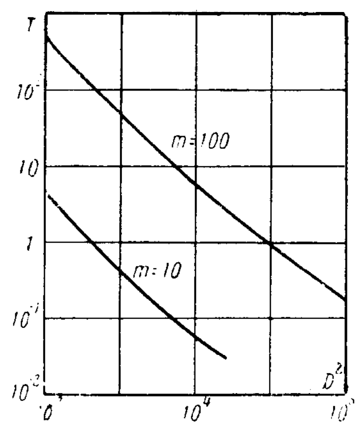

Fig. 7.

The problem of uniformly optimal system of damping, which would give the smallest mean square error in a large interval of time cannot be solved with mathematical tools used in this paper. Such a system would have variable parameters. As the first approximation for a program of variation of $\xi(\tau)$ we can take, for example, the previously obtained relationship $\xi(\tau) = \xi'(\tau)$. It is reasonable to expect that the errors of the system at any instant of time will be smaller than the errors resulting, for example, from Formula (3.6). A rigorous solution of this problem would be of great interest.

Let us repeat that our results are obtained from the most simplified model of the problem (1.1). Taking into account more complicated factors (rotation about the North-South axis, inertia of the gimbal suspension, the western velocity component, etc.) can somewhat change quantitatively the optimal parameters and increase errors of a gyrocompass. However, as seen from the graphs, the obtained relationships are quite "rough" in that they are not very sensitive to the change of parameters. We can expect, therefore, that the calculated errors which arise from the imperfection of the velocity meter add up, for all practical purposes to the errors arising from other causes.

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